Properties of sliced Wasserstein metric

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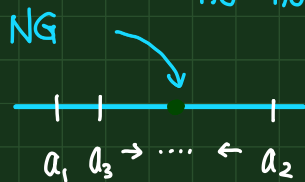
(X, d_X) : metric space $\rightsquigarrow$ $(\mathcal{P}_p(X), W_p)$: metric space $(p \in [1, \infty))$

(complete, separable) $\qquad$ (cplt, sep.)

"no hole" $\qquad$ "almost countable"



$^\forall \mu \in \mathcal{P}_p(X)$, $^\exists \left(\sum_{k=1}^{K} a_k \delta_{x_k} \right)_{K \in \mathbb{N}}$: approximates μ.

merit ☺ : • (X, d) : geodesic space $\Leftrightarrow$ $(\mathcal{P}_p(X), W_p)$: geod sp.

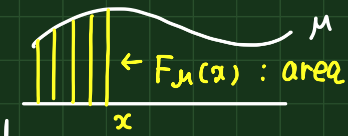
$^\forall x_0, x_1 \in X$: joined by a curve of length $d_X(x_0, x_1)$

• $^\exists$ dual problem.

$$W_p(\mu, \nu) = \inf \{ \|d_X\|_{L^p(\pi)} \mid \pi : \text{coupling of } \mu \,\&\, \nu \}$$

demerit ☹ • hard to compute W_p.

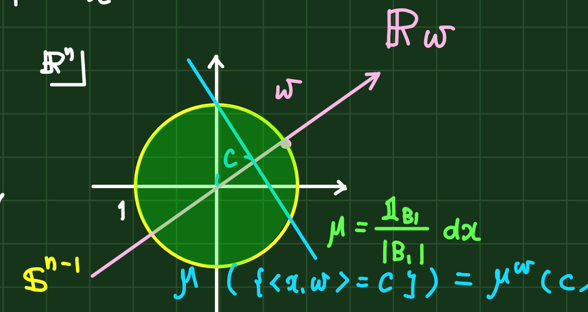
$X = \mathbb{R}$: easy $\quad F_\mu(x) := \mu((-\infty, x])$



$$W_p(\mu,\nu)^p = \int_{\mathbb{R}} |F_\mu^{-1}(x) - F_\nu^{-1}(x)|^p \, dx$$

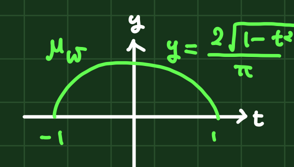
$\forall \mu \in \mathcal{P}_p(\mathbb{R}^n)$, $\forall \omega \in \mathbb{S}^{n-1}$

$$\mu^\omega((a,b]) := \mu(\{x \in \mathbb{R}^n \mid \langle x, \omega \rangle \in (a,b]\})$$



Def. $1 \le p < \infty$, $1 \le q \le \infty$

$$W_{p,q}(\mu,\nu) := \| W_p(\mu^\omega, \nu^\omega) \|_{L^q(\mathbb{S}^{n-1})} \text{ "="} \left(\int_{\mathbb{S}^{n-1}} W_p(\mu^\omega, \nu^\omega)^q \, d\sigma_{n-1}(\omega) \right)^{\frac{1}{q}}$$



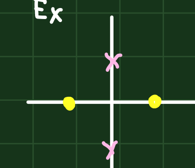
Thm. $(\mathcal{P}_p(\mathbb{R}^n), W_{p,q})$: complete, separable, metric space

(topologically equivalent to W_p)

dual prob ok for $p \le q$

geodesic for $n = 1$ or $p = 1$. NOT for $(n \ge 2 \ \& \ p > 1 \ \& \ q < \infty)$

Ex



$\mu = \frac{1}{2}(\delta_{e_1} + \delta_{-e_1})$

$\nu = \frac{1}{2}(\delta_{e_2} + \delta_{-e_2})$